Correction to Phase Boundary near $\nu=1$ in Forced van der Pol Oscillator

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Summary : The phase diagram of forced van der Pol oscillator near $\nu=1$ can not be explained by the traditional theory which assumes stationary solution such as quasi-periodic or fundamental oscillation. They are interpretted by the model that self-sustained oscillation is modulated by the external force. The applicability range of the theory is also discussed.

Key Words : Forced van der Pol Oscillator, Phase Diagram, and Approximation Method.

1. Introduction

Forced oscillations in nonlinear oscillator systems have been well investigated as fairly suitable theme of nonlinear oscillation theory[1]. In theoretical analyses appropriate stationary solutions are assumed such as fundamental (FO hereafter), quasi-periodic (QP hereafter), harmonic (HO hereafter) or self-sustained oscillations (SO hereafter), and their stabilities are analyzed. By these method the phase diagram of various oscillation modes is obtained. The example for forced van der Pol oscillator is represented in Fig.1 which is a well known diagram and is cited in usual textbooks[2]. Here the angular frequency of SO is taken to be 1. The vertical and horizontal axes represent the amplitude E and the angular frequency ν of external force. In the region written by "FO", "QP" and "HO" FO, QP and HO appear, respectively. The phase diagram shows that there is a critical value of E, E_c and for $E>E_c$ FO and for $E>E_c$ QP appear with fixed ν such as $\nu=2.5$.

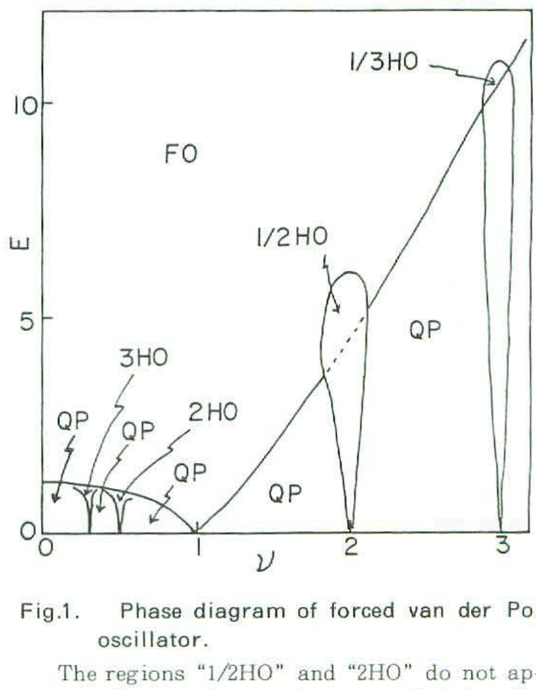

Fig.1. Phase diagram of forced van der Pol oscillator.

The regions "1/2HO" and "2HO" do not appear for eq. (1) since the asymmetry in nonlinearity is omitted.

Recently the detailed numerical analyses have been undertaken by the author for forced van der Pol oscillator[3]. His result shows that the region of $E<E_c$ where QP is expected to be present theoretically is not uniform and various oscillating states are present. Simultaneously transition phenomena near E_c have been reported which have not been predicted theoretically. As a sequel of this work we investigate the oscillation behaviours near $\nu\cong1$ of the forced van der Pol oscillator by numerical method. We find that the numerical result can not be explained by the traditional theory which assumes the presence of stationary solution such as FO or QP. This is caused by the fact that the assumption does not hold near $\nu\cong1$, and we succeed in deriving the model interpreting the result.

In the next section the method and the results of numerical analyses are mentioned. In section 3 the new idea which can explain the results is introduced. Section 4 is devoted to the concluding remarks and the applicability limit of the theory will be discussed.

2. Method and Results of Numerical Analyses

The basic equation studied in this article is the following forced van der Pol oscillator equation;

$$\ddot{x}-\mu(1-x^2)\dot{x}+\omega_0{}^2x=E\cos\nu t. \qquad (1)$$

μ and ω_0 are gain and angular frequency of SO and we take $\mu=1$ and $\omega_0=1$ without loss of generality. E and ν are amplitude and angular frequency of the external force. The equation is analyzed by second order Runge-Kutta method. To grasp the characteristics

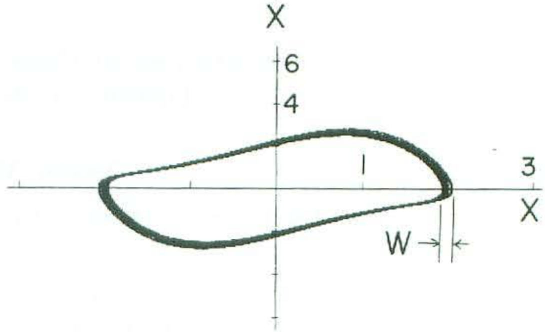

Fig.2. Phase space orbit for $E=0.13$ and $\nu=0.98$.

of the oscillations the orbit in phase space is utilized. The example is given in Fig.2. This is small E case where QP is expected to be present theoretically, however the circumference of the orbit resembles to that of free van der Pol oscillator $(E=0)$. Accordingly we regard that SO is modulated by the external force in this region of E. We take as a measure of modulation the quantity W defined in Fig.2. When $\nu=1.5$, W changes with E as shown in Fig.3[3]. There are three critical values of E, E_1, E_2 and E_3. For $E>E_3$ FO appears, that is, E_3 corresponds to E_c mentioned before. For $E<E_3$ QP is expected to be present, however this region is separated into three regions by E_1 and E_2. In the region of $E<E_1$ we can regard that SO is weakly modulated by the external force For $E_1<E<E_3$, SO and the external force. are comparable order. For $E_1<E<E_2$ SO prevails the external force slightly, and for $E_2<E<E_3$ *vice versa*. This is the result of ref. (3).

When ν approches 1 $(|\nu-1|\lesssim0.1)$ the region between E_1 and E_3 becomes narrow and E_1, E_2 and E_3 seem to be degenerate. Thus we set $E_1\cong E_2\cong E_3=E_c$. In this case W shows variation with E as represented in Fig.4. For E

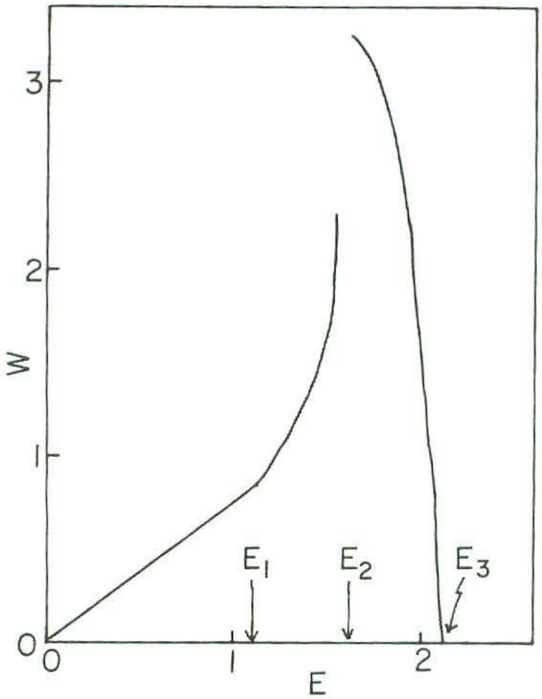

Fig.3. W *vs.* E for $\nu=1.5$.

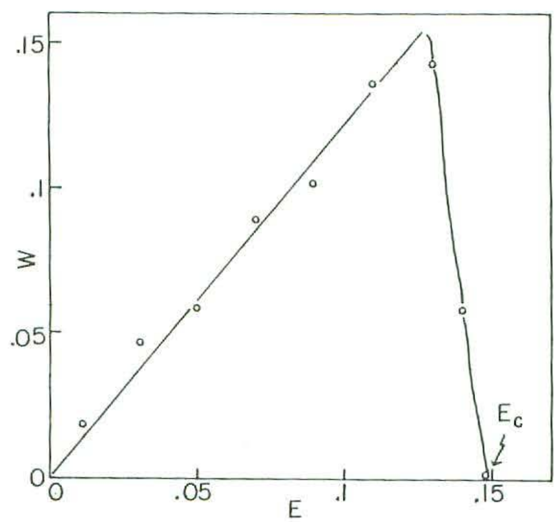

Fig.4. E-dependence of W for $\nu=0.98$.

$>E_c$ FO appears, and for $E<E_c$ QP comes

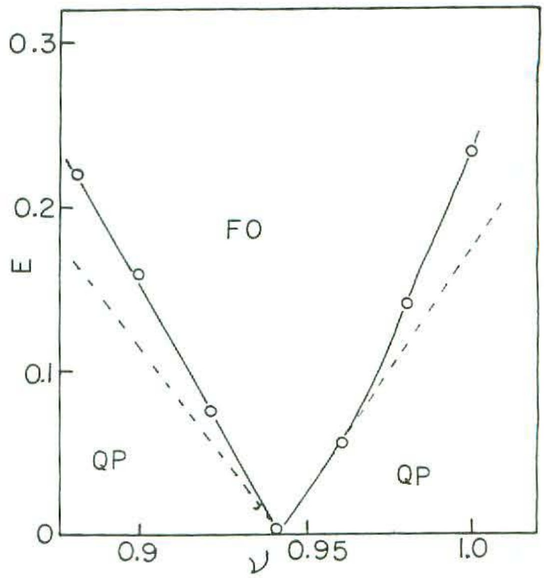

Fig.5. Numerically obtained phase boundary near $\nu\cong1$.

out. The ν-dependence of E_c is plotted in Fig.5. The dotted line in Fig.5 is a theoretical curve of E_c which will be discussed in section 3. This is a part of the phase diagram near $\nu\cong1$, however this diagram can not be explained by the traditional theory which assumes stationary solutions as will be discussed in the next section.

3. Theoretical Derivation of Phase Boundary near $\nu\cong1$

In the traditional theory the phase boundary or E_c separating the regions of FO and QP is derived by assuming the stationary solution of FO or QP in eq. (1).

At first we assume FO to eq. (1), that is,

$$x(t)=a\cos\nu t+b\sin\nu t. \tag{2}$$

Substituting eq. (2) into eq. (1) and equating the terms of $\cos\nu t$ and $\sin\nu t$, we obtain the following relation;

$$[(1-r^2/4)^2+\{(1-\nu^2)/\mu\nu\}^2]r^2 = (E/\mu\nu)^2. \quad (3)$$

Here $r^2=a^2+b^2$. By the stability analysis of this solution the condition $r \geq \sqrt{2}$ is found. E_c is determined from eq (3) by letting $r=\sqrt{2}$, which gives

$$E_c=\{2(1-\nu^2)^2+\mu^2\nu^2/2\}^{1/2}. \quad (4)$$

On the other hand if we assume QP solution to eq. (1)

$$x(t)=a\cos\nu t+b\sin\nu t+r_s\cos\nu_0 t. \quad (5)$$

The first and second terms represent the component of the external force and the third that of SO. Substituting eq. (5) into eq. (1) and equating the terms of $\cos\nu t$, $\sin\nu t$, $\cos\nu_0 t$ and $\sin\nu_0 t$ we obtain the following relation;

$$\nu_0=1 \text{ and } r_s^2+2r^2=4,$$
$$[(2-3R^2/4)^2+\{(1-\nu^2)/\mu\nu\}^2] \times(4-R^2)=(E/\mu\nu)^2. \quad (6)$$

Here $r^2=a^2+b^2$ and $R^2=r^2+r_s^2$. In this case E_c is decided when we set $r_s=0$ and $r=\sqrt{2}$ in eq. (6), since FO appears for $r_s=0$. Of course E_c obtained from eq. (6) coincides with eq. (4). E_c becomes minimum for $dE_c/d\nu=0$. From eq. (4) ν_{min} which gives lowest value of E_c is determined as,

$$\nu_{min}=(1-\mu^2/8)^{1/2}, \quad (7)$$

which is a reasonable result since eq. (7) gives for $\mu \ll 1$, $\nu_{min} \cong 1-\mu^2/16$. As is well known this is a corrected value of the angular frequency of SO to first approximation obtained by perturbation method. The value of E_c. $E_{c\,min}$ for $\nu=\nu_{min}$ is then obtained as,

$$E_{c\,min}=(17\mu^2/32-\mu^4/16)^{1/2}. \quad (8)$$

For $\mu=1$ $\nu_{min} \cong 0.94$ and $E_{c\,min} \cong 0.68$, however numerical result shows $E_c \cong 0$ for $\nu=0.94$ (Fig.5), which indicates the above discussions not to be correct.

We consider this discrepancy to originate from the assumption of the stationary oscillation such as FO or QP. Instead of them we use as the solution to eq. (1) SO modulated by the external force for small E, The form of the orbit in Fig.2 suggests this idea. Thus we assume the solution of eq. (1) as follows;

$$x(t)=\{A+a(t)\}\cos t. \quad (9)$$

Here A is the amplitude of SO and we consider that the external force causes modulation $a(t)$ in the amplitude. The relation $A \cong 2 \gg a(t)$ holds. Substituting eq. (9) into eq. (1) yields

$$\ddot{a}\cos t-\{2\sin t+(A+a)^2\cos^2 t\} \times(A+a)\sin t\dot{a}+(A+a)^3\cos^2 t\sin t = E\cos t\cos\{(\nu-1)t\} - E\sin t\sin\{\nu-1)t\}. \quad (10)$$

Since $a(t)$ is considered to be slowly varying, $\ddot{a}(t) \cong 0$. Using the relation $A \cong 2 \gg a(t)$ and equating the terms of cost and sint the following relation is obtained;

$$\dot{a}(t) \cong (E/2)\cos\{(\nu-1)t\}. \quad (11)$$

W is estimated nearly equal to be $2\sqrt{\langle a^2(t)\rangle}$ where the average $\langle \ \rangle$ is taken over a period. Thus eq. (11) gives,

$$W \cong E/\{\sqrt{2}(\nu-1)\}. \quad (12)$$

E_c may be determined from the condition that the relation $A \gg a(t)$ does not hold, that is, $W \cong A$. Thus E_c is decided from eq. (12) as,

$$E_c \cong \sqrt{2}(\nu-1)A \cong 2\sqrt{2}(\nu-1). \qquad (13)$$

The lowesest order correction to the angular frequency changes eq. (13) as,

$$E_c \cong 2\sqrt{2}(\nu-1+\mu^2/16)$$
$$\cong 2\sqrt{2}(\nu-0.94) \text{ for } \mu=1. \qquad (14)$$

Eq. (14) for $\mu=1$ is plotted in Fig.5 by dotted line. Although the quantitative agreement is incomplete, it explains the data qualitatively rather than the traditional theory does.

4. Concluding Remarks

It is found that the traditional theory can not explain the numerical results of the phase boundary of the forced van der Pol oscillator near $\nu \cong 1$. This fact indicates the failure of the assumption for small E that the stationary solutions such as QP and FO are present. Instead of it the model that SO is weakly modulated by the external force is proposed in small E region, which can explain the numerical results although it is incomplete quantitatively. One of the reasons for this incompleteness may arise from the condition $W \cong A$ utilized when E_c is determined. As is clear from Figs.2 and 4 this condition is not necessarily fulfilled.

The numerical data can be well explained by the traditional theory for the region $|\nu-1| \gtrsim 0.2$, on the other hand they can not be done for the region $|\nu-1| \lesssim 0.1$. Why is it? One of the plausible reasons is as follows: if $A \gtrsim E_c$ the assumption of stationary solution may be available near E_c. While for $A \gg E_c$ SO may be dominant even if the external force acts on the oscillator. Accordingly the condition that the traditional theory holds can be considered to be as,

$$E_c \cong 2\sqrt{2}|\nu-1| \gtrsim 1, \qquad (15)$$

which gives $|\nu-1| \gtrsim 0.3$. While the proposed model in this article is appropriate for the region $|\nu-1| \lesssim 0.3$. This is a reasonable result which gives the criterion whether the trasitional theory or present model is appropriate.

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外力に駆動されたファン・デァ・ポール振動子の $\nu=1$ 近傍の相図への補正

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要旨：$\nu=1$ 近傍での強制 van der Pol 振動子の相図は，定常的な概周期振動や基本調波振動を仮定する従来の理論では，うまく説明出来ない．然し自然振動が外力により弱く変調されるというモデルによって解釈されることが解った．そのモデルの適用範囲についても議論することができた．