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Frequency Shift of Self-Sustained Oscillation by External Force in Forced van der Pol Oscillator

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Summary: The orbital portrait of quasi-periodic oscillation (QP) generating in forced van der Pol oscillator changes by external force, as if a kind of frequency pulling effect takes place, which has been revealed by numerical analyses. QP consists of self-sustained and external force components. If the former frequency does not change by the latter, the portrait should not change remarkably. But traditional stationary solution analyses have not predicted such frequency change. This frequency change is calculated to the first order in the external force strength by extending the traditional theory. Thus the change of portraits can be explained by the frequency pulling effect among the two components of QP induced by this frequency change, and the transition points of QP portraits are roughly estimated.

Key Words: Forced van der Pol Oscillator, Quasi-Periodic Oscillation, Frequency Pulling Effect.

1. Introduction

When periodic external force is applied to self-sustained oscillator system, various kinds of oscillation modes are generated[1]. As an example, let us consider forced van der Pol oscillator of the form,

$$\frac{d^2x}{dt^2}-\mu(1-x^2)\frac{dx}{dt}+\omega_0{}^2x=E\cos\nu t. \qquad (1)$$

When E=0, the self-sustained oscillation (SO hereafter), $x(t)=2\cos(\omega t+\theta)$ is generated for $\mu>0$, where $\omega\cong\omega_0(1-\mu^2/16)$ and θ depends on the initial conditions. The RHS shows the periodic external force. The set of parameters, E and ν, decides the oscillation modes, and thus constitutes a phase diagram. For simplicity the case of $\omega_0=1$ and $\mu=1$ is taken in the following without loss of generality. The phase diagram is represented in Fig. 1. For $E>E_3(\cong\sqrt{2}\nu[\{(1-\nu^2)/\nu\}^2+1/4]^{1/2}$ [2]), the oscillation mode is synchronized to the external force, and the fundamental oscillation (FO hereafter) takes place. The region where the FO appears is designated as "FO" in Fig. 1. Typical orbital portrait (portrait hereafter) described in the phase space, $(x(t), \dot{x}(t))$-plane is represented in Fig. 2(c) for $E=7.7$ and $\nu=2.5$. The FO is shown to be nearly harmonic. On the other hand for $E<E_3$, two possibilities are present, one is the case, when the relation, $\nu\cong n\omega_0$ (n=1, 2, 3 …… or 1/2, 1/3, ……) is satisfied. In this case harmonic oscillation (HO hereafter) is generated, and the region is shown as "HO" in Fig. 1. Typical portrait is given in Fig. 2(a) when $E=10.0$ and $\nu=3$. It usually takes a form of simply closed curve. The other is the case, when above relation is

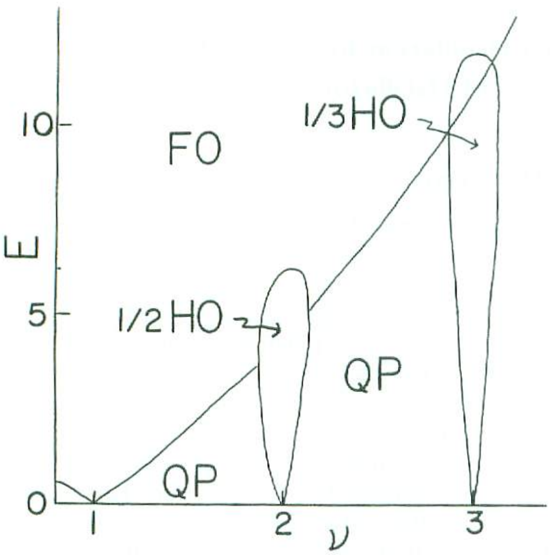

Fig. 1. The phase diagram of oscillation modes in forced van der Pol oscillator. "n" HO indicates n−(sub) harmonic oscillation mode.

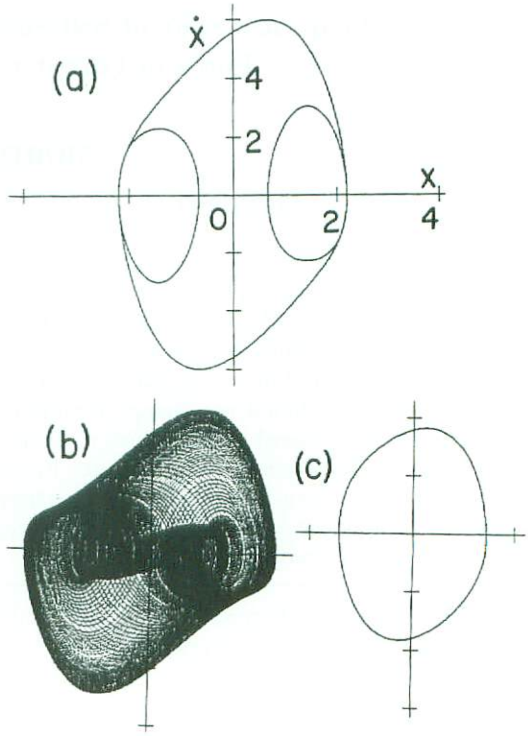

Fig. 2. The typical examples of orbital portraits for HO, QP and FO when (a)−ν=3, E =10.0, (b)−ν=2.33, E=4.0 and (c)−ν=2.5, E =7.7, respectively. The scale is represented in (a), and is common.

not satisfied, and quasi-periodic oscillation (QP hereafter) consisting of SO and external force components appears. Such region is designated as "QP" in Fig. 1, and the example of portrait is shown in Fig. 2(b) when E =4.0 and $\nu=2.33$. Usually the orbital points wander around in the limited region of phase space.

The portrait of FO does not show remarkable change even though E varies, because the frequency is completely synchronized to the external force, and E only affects its amplitude slightly. The HO consists of two com--ponents, one the SO and the other the external force as similarly as QP. However the frequency of SO is synchronized to that of external force due to the relation, $\nu \cong n\omega_0$ (frequency pulling effect), and can not be affected by E. Thus the portraitof HO is little influenced by E as similarly as the case of FO. On the contrary to above two cases, the portrait of QP is largely influenced by E, as if a kind of transition takes place, which has been revealed recently through the numerical analyses by the author[2]. The examples are given in Fig. 3 which shows the variations of portraits with E when ν is fixed to be 2.5. The details will be mentioned in the next section. The transition is very remarkable between the wandering type and simply closed type. The frequency pulling effect between two components of QP is necessary for such transition to take place. The frequen-

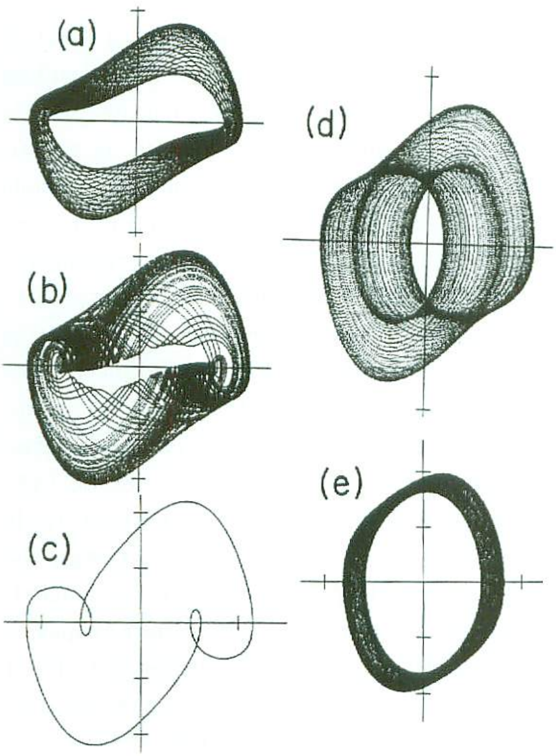

Fig. 3. The variations of portraits with E when $\nu=2.5$. The parameters are (a)E=1.5, (b) E =2.8, (c) E=3.4, (d) E=7.05 and (e) E=7.62. The scale aer same as those of Fig.2.

cy of SO should change by E, since that of the external force is fixed. But in the traditional stationary solution analyses, such frequency change of SO by E has not been taken into account, and region of QP has been regarded as a uniform region. But such a point of view can not explain the observed changes of QP portrairs.

We try in this article to take into account of the effect of E in first order on the frequency shift of SO, which has the possibility to induce the frequency pulling effect on the QP. Such effect might explain the observed transitions of QP portraits by E. As a result, the transition points are roughly estimated, although the quantitative agreements are poor. In the next section the changes of QP portraits with E are illustrated in detail. The aim of this article is to explain them. Simultaneously the frequency pulling effect in the model QP system[3] will be discussed, which is necessary to explain the changes of QP portraits. In section 3, the traditional theory are reviewed, and the central idea will be introduced concerning the frequency shift of SO by E. It is some extention of traditional theory. The last setion is devoted to the concluding remarks.

2. The Changes of QP Portraits by E and Frequency Pulling Effect in Model QP System

The changes of QP portraits by E are shown in Fig. 3 for the case of $\nu=2.5(\mu=1$ and $\omega_0=1)$ in eq.(1), which is obtained numerically through second order Runge-Kutta method[2]. When $E<E_1(=1.702)$, the portrait takes the form of Fig. 3(a) which consists of SO weakly modulated in its amplitude by the external forcs, and has been succesfully explained[4]. When E exceeds E_1, the portrait suddenly changes its form as given in Fig. 3(b) which is wandering type mentiond before. This is certainly the portrait of QP. But when E reaches $E_2(=3.261)$, it suddenly changes its form into that given in Fig. 3(c), as if a kind of transition takes place into the HO, that is, the simply closed type. Above fact says that some kind of synchronous state is attained in the two components of QP. This portrait continues up to $E=E_2'(=7.047)$. For $E_2<E<E_2'$ the frequency pulling effect seems to be realized between the SO and the exter-

nal force components. Such change has not been predicted in the traditional theory. For $E > E_2'$ the frequency pulling effect breaks down, and the portrait changes again into the wandering type as shown in Fig. 3(d). This portrait transforms at $E = E_3' (= 7.601)$ into the form given in Fig. 3(e) which finally changes continuously into that of FO at $E = E_3 (= 7.660)$. The portrait between E_3' and E_3(Fig. 3(e)) can be interpreted as the FO weakly modulated in its amplitude by the SO similarly as that below E_1. For $E > E_3$ the FO is realized, and the portrait has already been given in Fig. 2(c).

The region of interest is $E_1 < E < E_3'$. In the traditional theory this region is analyzed by assuming $x(t) = a\cos(\nu t + \theta) + b\cos(\omega t + \phi)$ and $\omega \cong 1 - \mu^2/16$, after which the stability of this solution is investigated. This stability analysis decides the region of QP in the phase diagram. Of course the first term is forced component, and the second the SO-component. ω is assumed to be constant not depending on E. Only the amplitudes, a and b, depend on E slightly. Thus the portrait should not be changed remarkably by E. But the numerical results indicate that above assumption breaks down, that is, ω should depend on E. If so, the frequency pulling between two components of QP may be possible, when the condition, $\nu \cong n\omega (n = 1, 2, 3 \cdots \cdots)$ is brought by E. This effect can change the portrait of QP drastically as observed numerically in Fig. 3. In the cases of FO and HO, the frequency pulling effect can not affect the portrait, because the effect has already been included.

Next let us see the frequency pulling effect in model QP system, $x(t) = r_1\cos\omega_1 t + r_2\cos\omega_2 t$[3]. Without loss of generality the parameters are taken as $r_1 = 1$, $r_2 = 1.5$ and $\omega_1 = 1$, and the portrait is investigated as a function of ω_2. As is expected the frequency pulling effect is obslrved, when the relation, $n\omega_1 \cong m\omega_2 (\omega_1 = 1$ and n, m are integer), is satisfied. When just $n\omega_1 = m\omega_2$, and n, m are simple integers such as 1, 2, 3, the portrait is simply closed curve, which is of course reasonable. But when $n\omega_1 \neq m\omega_2$, but $|n\omega_1 - m\omega_2| \lesssim 0.1$, new type oscilatory behaviour appears. It is n- or m-folded periodic oscillation of the point passing through the x-axis of phase space, $(x, \dot{x})$-plane, that is the point satisfying $\dot{x}(t) = 0$. For the details see ref.(4). It should be minded that the portrait itself is not simply closed curve type, although the frequency pulling effect seems to be induced. The regions of ω_2/ω_1, where such frequency pulling effect occurs, are represented in Fig.4 by thick bars. When n/m is simple rational number such as 1, 2, 3 or 1/2, 1/3, the region is wide($\cong 0.2$ in width of ω_2). On the other hand it is narrow($\lesssim 0.1$), when n/m is not so simple such as 3/4, 7/5 or 5/8. When the ratio, n/m is nearby irrational number such as $\sqrt{2} \cong 1.4142$ or $\sqrt{3} \cong 1.732$, the effect can hardly be observed. Two examples are showh by A and B in Fig. 4. To be interesting is that the Fig. 4 resembles to the "Devil's Staircase" appearing in the Cantor set[5].

3. Review of Traditional Theory and the Frequency Shift

At first let us review the case of $E = 0$ for later references. Since μ is usually utilized as a smallness parameter, we recover the variable, μ from $\mu = 1$. Then the solution, $x(t)$ of eq.(1) is expanded as, $x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots\cdots$. Simultaneously, the fre-

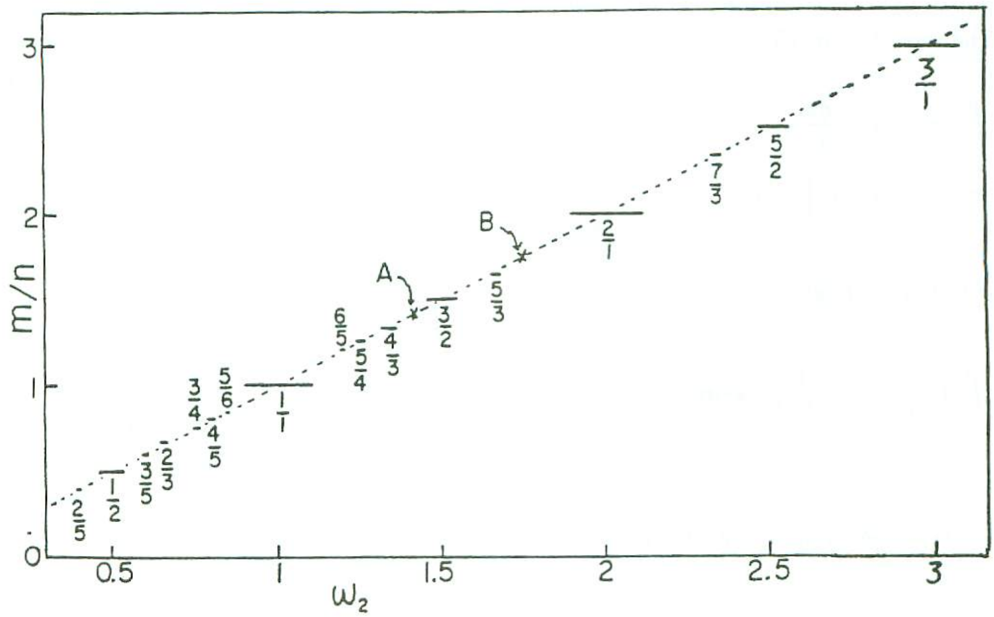

Fig. 4. The ranges of frequency pulling effect in model QP system. Dotted line shows $m/n=\omega_2/\omega_1$, and the thick bars indicates the regions of frequency pulling effect.

quency must be scaled so as to exclude the unphysical secular terms from the solution, so we let $\tau=\omega t$, and expand ω as $\omega=1+\mu\omega_1+\mu^2\omega_2+\cdots\cdots$. The initial conditions are taken as follows for convenience: $x_0(0)=A$(a constant to be determined later), $\dot{x}_0(0)=0$, $x_i(0)=0$ and $\dot{x}_i(0)=0$, $i=1, 2, 3\cdots\cdots$. Rewriting eq.(1) by $x_i(t)$ and τ and compaing the same order of μ, the following solution is obtained by recovering the original variables,

$$x(t)=(2-\mu^2/8)\cos\omega t+(3\mu/4)\sin\omega t +(3\mu^2/16)\cos 3\omega t-(\mu/4)\sin 3\omega t -(5\mu^2/96)\cos 5\omega t+\cdots\cdots, \quad (2)$$

and

$$\omega\cong 1-\mu^2/16. \quad (3)$$

Here ω_i's and the arbitrary constants appearing when the equations of any orders of μ are integrated, are determined so as to eliminate the secular(divergent) terms. The frequency, ω is shown to decrease as the components of higher harmonics increases, that is, as μ increases, which can be shown generally by the following method[6]. Multiplying $x(t)$ to eq.(1), and integrating (or averaging) over a period, $T(=2\pi/\omega)$ gives for the first term of LHS,

$$\frac{1}{T}\int_0^T x\frac{d^2x}{dt^2}dt=\frac{1}{T}x\frac{dx}{dt}\Big|_0^T -\frac{1}{T}\int_0^T\left(\frac{dx}{dt}\right)^2dt=-\frac{1}{T}\int_0^T\left(\frac{dx}{dt}\right)^2dt.$$

because of its periodicity, $x(0)=x(T)$ and $\dot{x}$

$(0)=\dot{x}(T)$. The partial integration is used once. The second term vanishes because,

$$\frac{1}{T}\int_0^T x(1-x^2)\frac{dx}{dt}dt = \frac{1}{T}(x^2/2-x^4/4)\Big|_0^T = 0.$$

Finally following relation appears,

$$-\frac{1}{T}\int_0^T \left(\frac{dx}{dt}\right)^2 dt + \frac{1}{T}\int_0^T x^2 dt = 0 \qquad (4)$$

Let us define the Fourier transform of $x(t)$ as,

$$x(t)=\sum_{n=1}^{\infty} a_n\cos(n\omega t+\alpha_n),\quad \omega=2\pi/T, \qquad (5)$$

and substitute this into eq.(4). Then we obtain,

$$\omega^2=\sum_{n=1}^{\infty} a_n^2 / \sum_{n=1}^{\infty} n^2 a_n^2 (\leqq 1). \qquad (6)$$

If no harmonic component is present, that is, $a_n=0$ for $n=2, 3, 4\cdots\cdots$, $\omega=1$ as is expected, that is, no frequency shift. Eq.(6) clearly indicates that the inclusion of higher harmonics induces the decrease in its frequency. From eq.(2) $a_1^2=(2-\mu^2/8)^2+(3\mu/4)^2 \cong 4+\mu^2/16$ and $a_3^2=(3\mu^2/16)^2+(\mu/4)^2 \cong \mu^2/16$ are obtained to the order of μ^2. If the terms up to $n=3$ are taken into account, eq.(6) gives for ω using a_1 and a_3,

$$\omega \cong \{(4+2\mu^2/16)/(4+10\mu^2/16)\}^{1/2} \cong 1-\mu^2/16,$$

which agrees with eq.(3), the result of reductive perturbation method. Above treatment is cited in the textbook[6]. In the following this method is extended to the case of $E\neq 0$ to obtain the E-dependence of ω. The similar procedure mentioned above for RHS of eq (1) gives,

$$\frac{E}{T}\int_0^T x(t)\cos\nu t\,dt = \frac{E}{2T}\sum_{n=1}^{\infty} a_n \times \int_0^T [\cos\{n\omega+\nu)t+\alpha_n\} + \cos\{(n\omega+\nu)t+\alpha_n\}]\,dt$$
$$= \frac{E}{2T}\sum_{n=1}^{\infty} a_n\Big[\frac{1}{n\omega+\nu}\{\sin\{(n\omega+\nu)T+\alpha_n\}-\sin\alpha_n\} + \frac{1}{n\omega-\nu}\times \{\sin\{(n\omega-\nu)T+\alpha_n\}-\sin\alpha_n\}\Big] \qquad (7)$$

Here let us take into account only of the dominant term which comes from the n^*-th term satisfying $n^*\omega-\nu\cong 0$. Then eq.(7) is simply approximated as, by letting $\delta=n^*\omega-\nu$,

$$(E/2T\delta)a_{n*}\{\sin(\delta T+\alpha_{n*})-\sin\alpha_{n*}\}.$$

Assuming $\delta T \ll 1$, $\{\cdots\cdots\}$ is approximated as $\delta T\cos\alpha_{n}*$. Eventually eq.(7) is reduced as $(E/2)a_{n*}\cos\alpha_{n*}$. Thus eq.(4) for $E\neq 0$ is modified as

$$-\omega^2\sum_{n=1}^{\infty} n^2 a_n^2 + \sum_{n=1}^{\infty} a_n^2 \cong a_{n*}E\cos\alpha_{n*},$$

or

$$\omega^2 \cong \left(\sum_{n=1}^{\infty} a_n^2 - a_{n*}E\cos\alpha_{n*}\right) / \sum_{n=1}^{\infty} n^2 a_n^2. \qquad (8)$$

This result gives the desired first order frequency shift by E, and is the main result of this article. Let us apply this result to the present problem. Since the unperturbed frequency is about $1-\mu^2/16(\mu=1)=0.9375$, n satisfying $n(1-\mu^2/16)-\nu=0(\nu=2.5)$ is 2.67. Thus n*, the integer closest to above n(= 2.67) is decided to be 3. Then from eq.(2),

$$\cos\alpha_{n*}=\cos\alpha_3=(3\mu^2/16)\{(3\mu^2/16)^2+\mu^2/16\}x^{1/2}\cong 0.688,$$

is obtained. Substituting this into eq.(8) and including up to the terms of n=3, the following relation is obtained,

$$\omega\cong 0.935-0.018E. \quad (9)$$

This is the final result. Since the value of E in problem is ranged from 3 to 7, the central value of ω is about 0.85. Thus the value of n satisfying $n\omega\cong\nu(\nu=2.5)$ is 3. The coincidence of this value with n* is accidental. As is clear from Fig.4 the range of ω_2/ω_1 is between 2.9 and 3.1, where the frequency pulling effect takes place for $\omega_2=3\omega_1$. Thus the effect should be realized in the region of E satisfying $\nu/\omega=2.9$ and 3.1, which gives two critical values of E, one 4.05, and the other 7.14. The frequency pulling effect takes place for $4.05<E<7.14$, where the portrait can be simply closed curve as if it were HO. It fails for $E<4.05$ and $E>7.14$, where the portrait has the form of wandering type. Comparing above results with the numerical results of Fig.3, E=4.05 and 7.14 can be considered to correspond to $E_2(=3.26)$ and $E_2'(=7.05)$, respectively. The relative errors are 24% and 1% for E_2 and E_2', respectively, which would largely depend on the accuracy of the frequency pulling range such as 2.9 or 3.1. Although the quantitatve agreement is poor, we should be satisfied considering the approximation to be the lowest order.

Thus we can explain the changes of QP portraits by the frequency pulling effect between the SO and the external force components, originating from the frequency shift of SO induced by the external force. This is the first assertion of frequency shift of SO by E as far as the author knows.

4. Concluding Remarks

For the QP generating in the forced van der Pol oscillator under weak external periodic force, the traditional theory predicts the ratio of SO-component to the external force to change by E, however it does not say the frequency change of SO by E. Thus the portrait of QP is expected to show little change by E. But the numerical analyses have revealed remarkble changes of QP portraits by E. Especially the portrait shows transition like behaviour, when it transforms from wandering type portrait to simply closed curve type with E. At those transition like points the interesting phenomena such as the metastable oscillation modes[2] have been reported. Such transformation of QP portraits can not be explained from the variation of amplitude ratio between the SO and the external force, but should originate from the frequency change of SO with E, because it has the possibility to induce the frequency pulling effect among the two components of QP. Such frequency shift has not been predicted by the traditional theory. To obtain the frequency shift of SO by E, eq.(1) or its modified equation is integrated(or averaged) over a period to investigate its DC-componont by substituting the Fourier series expansion of x(t) with multiples of basic frequency. Direct substitution of Fourier series into eq.(1) gives only trivial relation, because eq.(1)

is periodic and does not give the DC-component. Usually energy integral is obtained by multiplying dx(t)/dt to eq.(1). The system is periodc and is not conservative, and thus the energy integral does not give any meaningful relations as in the case of conservative system. So x(t) is multiplied to eq.(1), and the averaged equation is constructed, which gives meaningful result. Only the dominant term is taken into account for the external force term. The coefficients of Fourier series when no external force is present are utilized approximately. As a result, the frequency shift is estimated to the first order in the external force, and the transition points between wandering type and simply closed type portraits can be roughly estimated by comparing the frequency pulling effect in model QP systems with the estimated frequency shifts.

The problem left is the content of frequency pulling effect. In the frequency pulling range of model QP system, $2.9<\omega_2/\omega_1<3.0$, the portrait takes a form of simply closed curve only when $\omega_2/\omega_1=3$, but does not when $\omega_2/\omega_1\neq 3$. Instead of it, the time evolution of the point, x(t) satisfying $\dot{x}(t)=0$, shows the periodic motion, in which case the portrait is not always the simply closed curve. Thus the question is now left, whether the frequeney pulling effect can induce the observed transition between wandering type and simply closed curve type portraits, or not. But we should be satisfied at this stage with the fact that the possibility of frequency shift of SO is confirmed, which can cause the transition of QP portraits, and that the transition like points are roughly estimated.

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強制ファン・デァ・ポール振動子の自励振動の外力による周波数変化

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要旨：強制ファンデァ・ポール方程式における弱い外力のもとで発生する概周期振動は外力と共に変化することが数値解析により確められている．特に顕著なのはあたかも周波数引込みのような現象が発生することである．従来の定常解解析では信じられないことで，そこでは自励振動成分の周波数は変化せず，引き込み現象は発生し得えない．ここでは元方程式を適当に変形することによりＤＣ－成分を抽出し，それから自励振動の周波数が外力の振巾の一次で変化することが確められた．このことから，系に周波数引き込み現象の起ることが解り，同時に概周期振動が変化する外力の値を予言することができた．